Appendix 3.2

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1. **Extrema and local extrema.**

   Note that a maximum for a function may not be a *local* maximum. Consider $f(x) = x$ on $[0,1]$. Then the maximum occurs at $x = 1$. But this is **not** a local maximum. To be a local maximum there has to exist a $\delta > 0$ such that if $1 - \delta < x < 1 + \delta$ then $f(x) \le f(1)$. Yet this interval for x contains $x > 1$ for which $f(x)$ is **not** defined.

2. **Example 3.2.19** *Show that*

   $$\tan x = \frac{1}{x}$$

   has a solution in $(0, \pi/2)$. *Show that the solution is* ***unique*** *in that interval.*

   Solution Follow our principle of disliking fractions so multiply up and solve $x\tan x = 1$ for non-zero x. For this reason let $f(x) = x\tan x - 1$.

   Then $f(0) = -1 < 0$. Unfortunately, $f(\pi/2)$ is undefined. Instead try $f(x)$ at other values in $(0, \pi/2)$, hoping to try and find a sign change.

Try first $x = \pi/3$ since we know from the appropriate right angled triangle that $\tan(\pi/3) = \sqrt{3}$ so

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3}\sqrt{3} - 1 \geq \frac{3}{3}\sqrt{3} - 1 > 0.$$

Thus there is a zero c of f in $(0, \pi/3) \subseteq (0, \pi/2)$. Since $c > 0$, i.e. non-zero, we can divide to see that c is a solution of

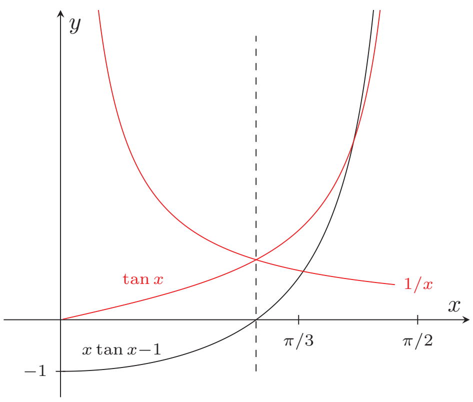
$$\tan c = \frac{1}{c}.$$

Assume for contradiction that f has more than one zero in $(0, \pi/2)$. Taking two such points c_1 and c_2, we have two points at which f has the same value (namely 0) and so we can apply Rolle's Theorem to f on $[c_1, c_2]$ to find a $c_3 : c_1 < c_3 < c_2$, between these two roots, for which $f'(c_3) = 0$. But

$$f'(x) = \tan x + \frac{x}{\cos^2 x} > 0$$

for $0 < x < \pi/2$. This contradiction means that f has only one zero in $(0, \pi/2)$. ■

On the graph we can see the intersection of $\tan x$ and $1/x$. We can also see the root of $x \tan x - 1$.



3. **Example** *Show that*

$$\sin x = x^3$$

has exactly three solutions in $[-\pi/2, \pi/2]$.

Solution Let $f(x) = \sin x - x^3$. In the notes we have found a $c \in (\pi/4, \pi/2)$ for which $f(c) = 0$. Note that

$$f(-c) = \sin(-c) - (-c)^3 = -\left(\sin c - c^3\right) = -f(c) = 0.$$

Finally $f(0) = 0$ and so we have found 3 real solutions of $\sin x = x^3$ in $[-\pi/2, \pi/2]$, namely $c, -c$ and 0.

Assume for contradiction that $f(x) = \sin x - x^3$ has **more** than three real roots. Looking at any four of these roots, called a_1, a_2, a_3 and a_4 say, we have

$$0 = f(a_1) = f(a_2) = f(a_3) = f(a_4).$$

We can apply Rolle's Theorem to each of the intervals $[a_1, a_2]$, $[a_2, a_3]$ and $[a_3, a_4]$, to find $c_1 \in (a_1, a_2)$, $c_2 \in (a_2, a_3)$ and $c_3 \in (a_3, a_4)$ with

$$0 = f'(c_1) = f'(c_2) = f'(c_3).$$

Continuing, $f'(x) = \cos x - 3x^2$ is differentiable on $\mathbb{R}$ and thus continuous on $\mathbb{R}$. So we can apply Rolle's Theorem to f' on the intervals $[c_1, c_2]$ and $[c_2, c_3]$ to find $d_1 \in (c_1, c_2)$ and $d_2 \in (c_2, c_3)$ with

$$0 = f^{(2)}(d_1) = f^{(2)}(d_2).$$

And again, $f^{(2)}(x) = -\sin x - 6x$ is differentiable on $\mathbb{R}$ and thus continuous on $\mathbb{R}$. So we can apply Rolle's Theorem to $f^{(2)}$ on the interval $[d_1, d_2]$ to find $e \in (d_1, d_2)$ for which $f^{(3)}(e) = 0$. **But** $f^{(3)}(x) = -\cos x - 6$ which is **never zero**.

This contradiction means that there are at most three zeros. From the first part this means there are exactly three solutions. ■

4. **Example 3.2.20** *Let $p(x) = x^3 + \alpha x + \beta$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha > 0$. Show that $p(x)$ has **exactly** one real root.*

   SolutionIf $x > 1$ then $x^2 + \alpha > 1$ and so

   $$p(x) = \left(x^2 + \alpha\right) x + \beta > x + \beta$$

   which is > 0 if $x > -\beta$. So choose b greater than $\max(1, -\beta)$ when $f(b) > 0$.

   If $x < 0$ write $x = -y$ with $y > 0$. Then $p(x) = -y^3 - \alpha y + \beta$, so

   $$-p(x) = y^3 + \alpha y - \beta = \left(y^2 + \alpha\right) y - \beta.$$

Again if $y > 1$ (i.e. $x < -1$) then $-p(x) > y - \beta$ which is > 0 if $y > -\beta$ (i.e. $x < \beta$). So choose a less than $\min(-1, \beta)$ when $p(a) < 0$.

If there is any zero of $p(x)$ it must lie in $[a, b]$ since $p(x) > 0$ for all $x > b$ and $p(x) < 0$ for all $x < a$. Apply the Intermediate Value Theorem on $[a, b]$ to find a $c \in (a, b)$ for which $p(c) = 0$.

Assume for contradiction that $p(x) = 0$ has at least two solutions. Then by Rolle's Theorem there exists c between two of these zeros that satisfies $p'(c) = 0$. That is, $3c^2 + \alpha = 0$. But $\alpha > 0$ and $c^2 \geq 0$ means $3c^2 + \alpha > 0$. This contradiction means the assumption is false, hence $x^3 + \alpha x + \beta$ has exactly one real root. ■

5. **Example 3.2.21** *Prove that*

$$x - \frac{x^2}{2} < \ln(1 + x)$$

for $x > 0$.

Solution Define

$$f(t) = \ln(1 + t) - t + \frac{t^2}{2},$$

for $t \geq 0$. Given $x > 0$ apply the Mean Value Theorem to f on $[0, x]$ to find $c \in (0, x)$ for which

$$\begin{aligned} f(x) - f(0) &= (x - 0) f'(c) = x\left(\frac{1}{1 + c} - 1 + c\right) \\ &= x\left(\frac{1 + (1 + c)(-1 + c)}{1 + c}\right) = \frac{xc^2}{1 + c}. \end{aligned}$$

This is easily seen to be > 0 hence $f(x) > f(0) = 0$ as required. ■

6. **Example 3.2.22** $x^e \leq e^x$ *for all* $x > 0$.

Solution Define

$$f(t) = t - e \ln t,$$

for $t > 0$. Let $x > 0$ be given.

There are two cases.

The First case is when $x > e$. Apply the Mean Value Theorem to f on the closed interval $[e, x]$ to find $e < c < x$ such that

$$f(x) - f(e) = \left(1 - \frac{e}{c}\right)(x - e) > 0.$$

Thus, since $f(e) = 0$,

$$x - e \ln x > e - e \ln e = 0.$$

Exponentiate both sides to deduce $e^x > x^e$.

The second case is when $0 < x < e$. This time apply the Mean Value Theorem to f on the closed interval $[x, e]$ to find $0 < c < e$ such that

$$f(x) - f(e) = \left(1 - \frac{e}{c}\right)(x - e).$$

This time both factors $1 - e/c$ and $x - e$ are negative, but their product is still positive so again $f(x) - f(e) > 0$. Thus we again deduce $e^x > x^e$.

Finally, if $x = e$ the result follows with equality. ■

7. **Example 3.2.23**

$$e^x > 1 + x + \frac{x^2}{2} \text{ if } x > 0 \quad \text{and} \quad e^x < 1 + x + \frac{x^2}{2} \text{ if } x < 0.$$

Solution Define

$$f(t) = e^t - \left(1 + t + \frac{t^2}{2}\right).$$

The function f is continuous and differentiable on $\mathbb{R}$. Given $x \in \mathbb{R}$ apply the Mean Value Theorem to f on $[0, x]$ to find $c \in (0, x)$ such that

$$f(x) = f(x) - f(0) = f'(c)\, x = (e^c - 1 - c)\, x.$$

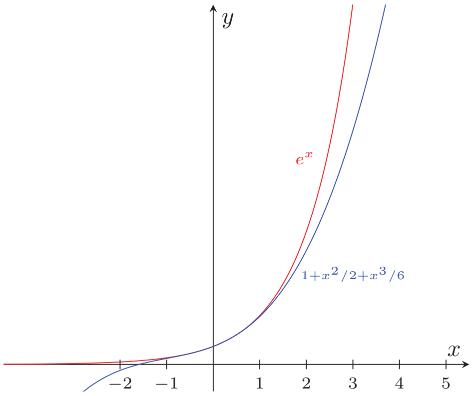
By Example 3.2.10 we have $e^c - 1 - c \geq 0$ for all c. Hence $f(x) \geq 0$ if $x \geq 0$, $f(x) \leq 0$ if $x < 0$. This is the required result. ■

8. **Example 3.2.24**

$$e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

for all $x \in \mathbb{R}$?

The following diagram suggests this is true:



Solution Define

$$f(t) = e^t - \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right).$$

The function f is continuous and differentiable on $\mathbb{R}$. Given $x \in \mathbb{R}$ apply the Mean Value Theorem to f on $[0, x]$ to find $c \in (0, x)$ such that

$$f(x) = f(x) - f(0) = f'(c)\,x = \left(e^c - 1 - c - \frac{c^2}{2}\right)x. \qquad (5)$$

We can use Example 3.2.24 on the first factor on the right hand side of (6) to see that both factors are either negative or both positive. In this way we see that $f(x) \geq 0$ for all x. This is the required result. ■

We can continue comparing e^x with the first terms from the power series definition. Can you see a pattern? We will return to this is a section on Taylor Series.

9. **Example 3.2.25**

$$\frac{1}{1 - x + \frac{x^2}{3}} > e^x > \frac{1}{1 - x + \frac{x^2}{2}},$$

the first inequality for $0 < x < 1$, *the second for all* $x > 0$.

Solution We only look at the left hand inequality here. On multiplying up it suffices to prove

$$1 > e^x \left(1 - x + \frac{x^2}{3}\right)$$

for $0 < x \leq 1$ and

$$e^x \left(1 - x + \frac{x^2}{2}\right) > 1$$

for $x > 0$.

i. Define

$$f(t) = 1 - e^t \left(1 - t + \frac{t^2}{3}\right).$$

It suffices to show that $f(x) > 0$ for $0 < x \leq 1$. Given $x : 0 < x \leq 1$ apply the Mean Value Theorem to f on the interval $[0, x]$ to find $c : 0 < c < x \leq 1$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c).$$

But

$$f'(t) = -e^t \left(1 - t + \frac{t^2}{3}\right) - e^t \left(-1 + \frac{2}{3}t\right) = \frac{1}{3}e^t t (1 - t).$$

Hence

$$f(x) - f(0) = f'(c)\, x = \frac{1}{3}e^c c (1 - c)\, x > 0,$$

since $x > 0$ and $1 > c > 0$. Yet $f(0) = 0$ so the result follows.. ■

10. **Example 3.2.26** *Define*

$$f_2(\theta) = \begin{cases} \dfrac{\sin\theta}{\theta} & \textit{if } \theta \neq 0 \\ 1 & \textit{if } \theta = 0. \end{cases}$$

Prove that f_2 is differentiable on $\mathbb{R}$.

In the Appendix to the first Section on Continuity it was shown that f_2 is continuous on $\mathbb{R}$. In the Appendix to the first Section on Differentiation it was noted that f_2 is differentiable iff

$$\lim_{\theta \to 0} \frac{f_2(\theta) - f_2(0)}{\theta - 0} = \lim_{\theta \to 0} \frac{\frac{\sin\theta}{\theta} - 1}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta - \theta}{\theta^2}$$

exists. But there is no elementary way to evaluate this limit so we had to wait until now when we can apply L'Hôpital's Rule,

$$\lim_{\theta \to 0} \frac{\sin\theta - \theta}{\theta^2} = \lim_{\theta \to 0} \frac{\cos\theta - 1}{2\theta}.$$

You could apply L'Hôpital's Rule a second time or note that we have seen this limit before, in the section on Special Limits, when it was shown to be 0. Hence f_2 is differentiable at 0 with derivative 0. For $\theta \neq 0$, $f_2(\theta) = \sin\theta/\theta$, a quotient of differentiable functions and so differentiable for all $\theta \neq 0$. ■

In fact, the derivative is

$$f_2'(\theta) = \begin{cases} \dfrac{\theta\cos\theta - \sin\theta}{\theta^2} & \text{if } \theta \neq 0 \\ 0 & \text{if } \theta = 0. \end{cases}$$

Can you show that this is differentiable on $\mathbb{R}$?

11. **Theorem 3.2.27** ***Increasing-Decreasing Theorem*** *Assume that f is differentiable on (a, b) and continuous on $[a, b]$.*

1) If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing on $[a, b]$.

2) If $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly decreasing on $[a, b]$.

3) If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$.

Proof Let $a \leq w < y \leq b$ be given. The Mean Value Theorem gives

$$f(y) - f(w) = f'(c)(y - w)$$

for some $w < c < y$.

1) If $f'(x) > 0$ for all $x \in (a, b)$ we have $f(y) - f(w) > 0$, i.e. $f(y) > f(w)$. Thus f is strictly increasing.

2) If $f'(x) < 0$ for all $x \in (a, b)$ we have $f(y) - f(w) < 0$, i.e. $f(y) < f(w)$. Thus f is strictly decreasing.

3) If $f'(x) = 0$ for all $x \in (a, b)$ we have $f(y) - f(w) = 0$, i.e. $f(y) = f(w)$. In particular, taking $w = a$ we see that $f(y) = f(a)$ for all $y \in [a, b]$. Hence f is constant. ■

12. An example of the use of L'Hôpital's Rule is that

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.$$

Recall that $\exp(x)$ is continuous on $\mathbb{R}$ so

$$\lim_{x\to 0} \exp\left(\frac{\ln(1+x)}{x}\right) = \exp\left(\lim_{x\to 0} \frac{\ln(1+x)}{x}\right).$$

That is,

$$\lim_{x\to 0} (1+x)^{1/x} = \exp(1) = e. \tag{6}$$

Next recall the alternative definition of a limit: For $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$,

$$\lim_{x\to a} f(x) = L$$

iff for all sequences $\{x_n\}_{n\geq 1}$ for which $x_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = a$ we have

$$\lim_{n\to\infty} f(x_n) = L.$$

In the case of (7) we choose the sequence $x_n = 1/n$ for $n \geq 1$ to deduce that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{1/n} = e.$$

That this sequence converges was shown in MATH10242.

13. **Be Careful!** L'Hôpital's Rule would immediately give

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0} \frac{\cos\theta}{1} = 1, \tag{7}$$

but **only** because we know that $\frac{d}{d\theta}\sin\theta = \cos\theta$. Yet if you look back at the example where we evaluated the derivative of $\sin\theta$ you see that we needed to know the value of the limit in (8). Thus it would be a circular argument to try to use the derivative in the evaluation of the limit!

L'Hôpital's Rule would also give

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta^2} = -\frac{1}{2}.$$

Yet in the Question sheets we see a method where this is deduced from $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$ *without* the need for differentiation. We should always show preference for the method based on the simplest concepts.

14. Further examples

Example 3.2.28 *Show that*

$$\cos x = \sinh x$$

*has **exactly one** solution in* $[0, \pi/2]$.

Solution in Tutorial Let $f(x) = \cos x - \sinh x$. Then

$$\begin{aligned} f(0) &= 1 - 0 > 0, \\ f(\pi/2) &= 0 - \sinh(\pi/2) \approx -2.30129... < 0. \end{aligned}$$

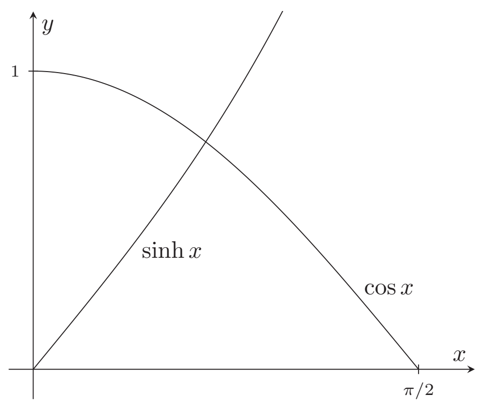
So, by the Intermediate Value Theorem, there exists $c \in (0, \pi/2)$ for which $f(c) = 0$.

Assume for a contradiction that there are at **least two** different solutions in (the open interval) $(0, \pi/2)$. Let c_1 and c_2 be two such values for which $f(c_1) = f(c_2) = 0$.

Then by Rolle's Theorem applied to f on $[c_1, c_2]$ there exists $c \in (c_1, c_2) \subseteq (0, \pi/2)$ for which $f'(c) = 0$.

Yet $f'(x) = -\sin x - \cosh x$. On $(0, \pi/2)$ we have $\sin x > 0$ and $\cosh x > 1$ and thus $f'(x) < -1$. This contradicts the claim that $f'(c) = 0$ for some $c \in (0, \pi/2)$. Therefore the last assumption must be false, and hence the solution in $(0, \pi/2)$ is unique. ■

The illustration of Example 3.2.29 is:



Example 3.2.29 *Show that*

$$\sin x = x^3$$

has ***exactly one*** *solution in* $[\pi/4,\ \pi/2]$.

Solution in Tutorial Let $f(x) = \sin x - x^3$. Then

$$f\left(\frac{\pi}{2}\right) = 1 - \left(\frac{\pi}{2}\right)^3 \approx -2.8757 < 0$$

while

$$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \left(\frac{\pi}{4}\right)^3 \approx 0.2226 > 0.$$

So, by the Intermediate Value Theorem, there exists $c \in (\pi/4,\ \pi/2)$ for which $f(c) = 0$.

Assume for a contradiction that there are at **least two** different solutions in $[\pi/4,\ \pi/2]$. Let c_1 and c_2 be two such values for which $f(c_1) = f(c_2) = 0$.

Then by Rolle's Theorem applied to f on $[c_1,\ c_2]$ there exists $c_3 \in (c_1, c_2) \subseteq [\pi/4,\ \pi/2]$ for which $f'(c_3) = 0$.

Yet $f'(x) = \cos x - 3x^2$. We will in fact show this is negative on $[\pi/4,\ \pi/2]$ by bounding it above by a negative number. It suffices to note that $\cos x < 1$ while $\pi/4 \le x \le \pi/2$ implies $x \ge \pi/4 > 3/4$ and thus $-x^2 \le -9/16$. Combine to get

$$f'(x) \le 1 - 3\frac{9}{16} = -\frac{11}{16} < 0$$

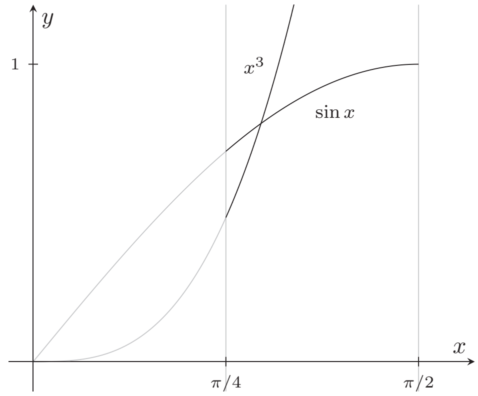
on $[\pi/4,\ \pi/2]$. This contradicts the claim that $f'(c_3) = 0$ for some $c_3 \in [\pi/4,\ \pi/2]$. Therefore the last assumption must be false, and thus the solution in $[\pi/4,\ \pi/2]$ is unique. ■

The illustration of Example 3.2.30 is:

From the series definition of e^x we have, for $x > 0$,

$$e^x = 1 + x + \frac{x^2}{2} + ... > 1 + x,$$

since the discarded terms are positive. This can be extended to negative x:



Example 3.2.30

$$e^x \geq 1 + x,$$

for all $x \in \mathbb{R}$ with equality when $x = 0$.

Solution Define $F(t) = e^t - 1 - t$. Given $x \in \mathbb{R}$ apply the Mean Value Theorem to F over the closed interval

$$\begin{cases} [x,\, 0] & \text{if } x < 0 \\ [0,\, x] & \text{if } x \geq 0. \end{cases}$$

We have then, for some c between x and 0, that

$$e^x - 1 - x - 0 = F(x) - F(0) = F'(c)\,(x - 0) = (e^c - 1)\,x. \quad (8)$$

There are two cases.

The first case is when $x \geq 0$. If $x = 0$ the required result follows with equality. If $x > 0$ then c satisfies $x > c > 0$, i.e. $c > 0$ in which case $e^c > 1$. Thus we have both $e^c - 1 > 0$ and $x > 0$, in which case $(e^c - 1)\,x > 0$ and from (2) we deduce that $e^x - 1 - x > 0$ as required.

The second case is when $x < 0$. This time $c < 0$ in which case $e^c < 1$. Then we have both $e^c - 1 < 0$ and $x < 0$ are negative. But the product of two negative numbers is positive so $(e^c - 1)\,x > 0$. Thus (2) again gives $e^x - 1 - x > 0$.

In both cases the required result follows. ■